EVOLUTION EQUATIONS ON NON FLAT WAVEGUIDES

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Abstract. We investigate the dispersive properties of evolution equations on waveguides with a non flat shape. More precisely we consider an operator

$$H = -\Delta_x - \Delta_y + V(x,y)$$

with Dirichled boundary condition on an unbounded domain Ω, and we introduce the notion of a *repulsive waveguide* along the direction of the first group of variables x. If Ω is a repulsive waveguide, we prove a sharp estimate for the Helmholtz equation $Hu - \lambda u = f$. As consequences we prove smoothing estimates for the Schrödinger and wave equations associated to H, and Strichartz estimates for the Schrödinger equation. Additionally, we deduce that the operator H does not admit eigenvalues.

1. Introduction

A *flat waveguide* is a domain Ω in $\mathbb{R}^{n+m}$ which can be written as a product of a bounded open subset ω with $\mathbb{R}^n$:

$$\omega \subseteq \mathbb{R}^m, \qquad \Omega = \mathbb{R}^n \times \omega \subseteq \mathbb{R}^n_x \times \mathbb{R}^m_y, \qquad n, m \geq 1.$$

Throughout the paper we shall denote with x the group of the first n variables and with y the last m variables in $\mathbb{R}^{n+m}$. Waveguides appear in many concrete applications, since they can be used to model various interesting physical structures such as *wires* and *plates* (see Figure 1). The Laplace operator on Ω with Dirichlet

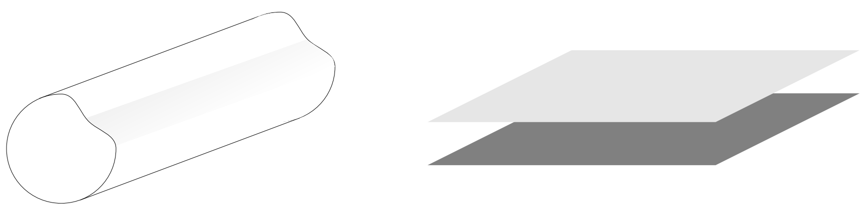

Figure 1. (a) $n = 1$, $m = 2$; (b) $n = 2$, $m = 1$

or Neumann boundary conditions has a natural splitting

$$\Delta_{x,y} = \Delta_x + \Delta_y$$

where Δ_x is the free Laplacian on $\mathbb{R}^n$ and Δ_y is the Dirichlet resp. Neumann Laplacian on Ω (we shall also write

$$\nabla = (\nabla_x, \nabla_y)$$

with obvious meaning). Thus the operator has a simple spectral structure: indeed, if we choose an orthonormal set of eigenfunctions $\{\phi_j(y)\}_{j\geq 1}$ for $-\Delta_y$ on ω and

Date: October 6, 2010.
2000 *Mathematics Subject Classification.* 35L70, 58J45.
Key words and phrases. waveguides, resolvent estimates, Strichartz estimates, Schrödinger equation .

denote by λ_j^2 the corresponding eigenvalues, the operator $-\Delta_{x,y}$ is equivalent to the sequence of operators on $\mathbb{R}^n$

$$-\Delta_x + \lambda_j^2.$$

As a consequence, the study of linear and nonlinear evolution equations on flat waveguides is quite similar to the standard case of free equations on $\mathbb{R}^n$. The theory was initiated in [11] and developed in [13] and [12].

Despite the simplicity of the theory, it is clear that the flatness assumption on the domain is not always realistic. Thus a natural question is whether a similar theory can be developed for more general, non flat waveguides. Here we begin to address this question, by investigating the smoothing and dispersive properties of wave and Schrödinger equations in more general situations. Such properties, which are usually expressed as global in time estimates on solutions of the linear equations, are the key ingredients for the nonlinear theory. To the best of our knowledge, the results in the present paper are the first ones concerning dispersive phenomena on non flat waveguides.

We start with a quick overview of the dispersive properties for the linear Schrödinger and wave-Klein-Gordon equations in the flat case.

Example 1.1. Consider the Schrödinger equation

$$iu_t - \Delta u = 0, \qquad u(0,x,y) = f(x,y) \tag{1.1}$$

with Dirichlet boundary conditions on $\Omega = \mathbb{R}^n \times \omega$, with ω a bounded open set in $\mathbb{R}^m$. Let ϕ_j, λ_j^2 be as above, then by expanding

$$u = \sum_{j\geq 1} u_j(t,x)\phi_j(y), \qquad f = \sum_{j\geq 1} f_j(x)\phi_j(y)$$

we can rewrite equation (1.1) as the equivalent family of independent equations

$$i\partial_t u_j - \Delta_x u_j + \lambda_j^2 u_j = 0, \qquad u_j(0,x) = f_j(x). \tag{1.2}$$

The term $\lambda_j^2 u$ can be absorbed in iu_t via the gauge transformation $u_j \to e^{i\lambda_j^2 t}u_j$, leaving the L^p norm of the solution unchanged. Thus from the explicit representation of the solution we have the *dispersive estimates*

$$\|u_j(t)\|_{L^\infty(\mathbb{R}^n)} \leq |t|^{-n/2}\|f_j\|_{L^1(\mathbb{R}^n)} \tag{1.3}$$

and summing over j we obtain

$$\|u(t)\|_{L^\infty(\Omega)} \leq |t|^{-n/2}\sum_{j\geq 1}\|\phi_j\|_{L^\infty(\omega)}\|f_j\|_{L^1(\mathbb{R}^n)} \equiv |t|^{-n/2}\|f\|_Z. \tag{1.4}$$

A more explicit expression of the norm $\|f\|_Z$ requires some information on the growth of the maximum norm of eigenfunctions. Typically one has

$$\|\phi_j\|_{L^\infty(\omega)} \lesssim \lambda_j^\sigma$$

for some $\sigma > 0$, and this leads to a dispersive estimate of the form

$$\|u(t)\|_{L^\infty(\Omega)} \lesssim |t|^{-n/2}\|(1-\Delta_y)^{\sigma/2+\epsilon}f\|_{L^1_x L^2_y(\omega)} \tag{1.5}$$

The pointwise estimate (1.5) is quite strong and we shall not be able to prove an analogous in the non flat case. However Schrödinger equations satisfy weaker but more general estimates called *Strichartz estimates*, which can be extended to our situation. Consider for maximum generality the nonhomogeneous equation

$$iu_t - \Delta u = F(t,x,y), \qquad u(0,x,y) = f(x,y) \tag{1.6}$$

with Dirichlet boundary conditions on Ω as above. Here we assume for simplicity $n \geq 3$. Expanding again

$$F = \sum_{j \geq 1} F_j(t,x)\phi_j(y)$$

we are led to the equations

$$i\partial_t u_j - \Delta_x u_j + \lambda_j^2 u_j = F_j(t,x), \qquad u_j(0,x) = f_j(x). \tag{1.7}$$

The endpoint Strichartz estimate (see [7], [10]) for u_j states that

$$\|u_j\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \lesssim \|f_j\|_{L_x^2} + \|F_j\|_{L_t^2 L_x^{\frac{2n}{n+2}}} \tag{1.8}$$

with constants independent of j. Squaring and summing over j we obtain the endpoint Strichartz estimate for flat waveguides:

$$\|u\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}} \lesssim \|f\|_{L^2_{x,y}(\Omega)} + \|F\|_{L^2_{t L_y^2} L_x^{\frac{2n}{n+2}}}. \tag{1.9}$$

We write the estimate in operator form as follows, where $\Delta = \Delta_{x,y}$ with Dirichlet b.c. on Ω, $n \geq 3$:
(1.10)

$$\|e^{it\Delta}f\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}} \lesssim \|f\|_{L^2(\Omega)}, \qquad \left\| \int_0^t e^{i(t-s)\Delta}F(s)ds \right\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}} \lesssim \|F\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n+2}}}.$$

Similar estimates hold when $n = 1, 2$.

An even weaker and more general form of estimates are the *smoothing estimates*, which go back at least to [9], see also [3]. For equations (1.7) they take the form

$$\|\langle x\rangle^{-1/2-\epsilon}|D_x|^{1/2}u_j\|_{L_t^2 L_x^2} \lesssim \|f_j\|_{L^2(\mathbb{R}^n)} + \|\langle x\rangle^{1/2+\epsilon}|D_x|^{-1/2}F_j\|_{L_t^2 L_x^2} \tag{1.11}$$

where we are using the notations

$$|D_x| = (-\Delta_x)^{1/2}, \qquad \langle x\rangle = (1+|x|^2)^{1/2}.$$

Squaring and summing over j we obtain

$$\|\langle x\rangle^{-1/2-\epsilon}|D_x|^{1/2}u\|_{L_t^2 L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|\langle x\rangle^{1/2+\epsilon}|D_x|^{-1/2}F\|_{L_t^2 L^2(\Omega)}. \tag{1.12}$$

Example 1.2. Consider the wave-Klein-Gordon equation for $u = u(t,x,y)$

$$u_{tt} - \Delta_{x,y}u + m^2 u = 0, \qquad m \geq 0, \qquad (x,y) \in \Omega = \mathbb{R}^n \times \omega \tag{1.13}$$

with Dirichlet boundary conditions. Proceeding as above we obtain the family of problems on $\mathbb{R}^n$

$$\partial_t^2 u_j - \Delta_x u_j + (\lambda_j^2 + m^2)u_j = 0. \tag{1.14}$$

Notice that in the case of Dirichlet b.c., even if we start from a wave equation for u (i.e. $m = 0$), the equations for u_j will always be of Klein-Gordon type since $\lambda_j^2 > 0$ for all j. Now, sharp dispersive estimates are known for the free equations (1.14), and summing over j we shall obtain dispersive estimates for the original equation (1.13). Indeed, using the notations $\langle D\rangle = (1-\Delta)^2$, $\langle D\rangle_M = (M^2 - \Delta)^{1/2}$, we can represent the solution of $\Box v + M^2 v = 0$ on $\mathbb{R}_x^n$ as

$$v(t,x) = \cos(t\langle D\rangle_M)v(0) + \frac{\sin(t\langle D\rangle_M)}{\langle D\rangle_M}v_t(0),$$

thus we see that the solution can be expressed via the operator $e^{it\langle D\rangle_M}$. To prove a dispersive estimate for it, we may use the following estimate in terms of Besov spaces

$$\|e^{it\langle D\rangle}f\|_{L_x^\infty} \leq \frac{C}{|t|^{n/2}}\|f\|_{B_{1,1}^{\frac{n}{2}+1}}$$

(see e.g. the Appendix of [5]), and by the scaling $v(t,x) \to v(Mt,Mx)$ we obtain

$$\|e^{it\langle D\rangle_M} f\|_{L^\infty_x} \le C \frac{M^{\frac{n}{2}}}{|t|^{n/2}} \|f(M\cdot)\|_{B^{\frac{n}{2}+1}_{1,1}}$$

The Besov norm in the estimate is not homogeneous, however at least for $M \ge c_0 > 0$ we get

$$|e^{it\langle D\rangle_M} f\|_{L^\infty_x} \le C(c_0) \frac{M^{n+1}}{|t|^{n/2}} \|f\|_{B^{\frac{n}{2}+1}_{1,1}}. \tag{1.15}$$

We can now apply this estimate to equation (1.13) i.e. to the sequence of problems (1.14). The relevant operator for (1.13) is

$$e^{it(m^2-\Delta_{x,y})^{1/2}} f = \sum_{j\ge 1} e^{it\langle D\rangle_{M_j}} f_j(x)\phi_j(y), \qquad M_j^2 = m^2 + \lambda_j^2$$

where of course $f(x,y) = \sum f_j(x)\phi_j(y)$. We obtain

$$\|e^{it(m^2-\Delta_{x,y})^{1/2}} f\|_{L^\infty_{x,y}} \le C|t|^{-n/2} \sum_{j\ge 1} (m^2+\lambda_j^2)^{\frac{n+1}{2}} \|f_j(x)\|_{B^{\frac{n}{2}+1}_{1,1}} \|\phi_j\|_{L^\infty}.$$

The last sum defines a norm of the initial data f which can be estimated by the $W^{N,1}$ norm of f for N large enough. See [11], [13] for more details and the applications to nonlinear wave equations. Following the same lines, one can prove Strichartz estimates for the Wave-Klein-Gordon equation on Ω.

Finally, smoothing estimates for the operators $e^{it\langle D\rangle_M}$ connected to the equation on Ω

$$u_{tt} - \Delta_{x,y} u + M^2 u = 0$$

take the form

$$\|\langle x\rangle^{-1/2-\epsilon} e^{it\langle D\rangle_M} f\|_{L^2_t L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{1.16}$$

The above approach, based on splitting and diagonalizing part of the operator, requires the domain to be of product type and breaks down for more general domains. Even the spectral problem is difficult, as the following considerations suggest.

Remark 1.1. For flat waveguides we have a purely continuous spectrum, also for *certain* locally perturbed waveguides, in particular for any local perturbation Ω of $(0,1)\times\mathbb{R}^{n-1}$, for which $\nu(x)\cdot x' \le 0$ holds for any $x = (x_1, x')$ on the boundary $\partial\Omega$, see construct local perturbations where the Dirichlet Laplacian has eigenvalues below its essential spectrum. But there may also exist eigenvalues embedded into the essential spectrum; see e.g. [21], where the following example is constructed. Let $D \subset \mathbb{R}^2$ be bounded, star-shaped with respect to the origin and invariant under the orthogonal group. Let $\rho \in C^0(\mathbb{R}^k)$ be positive, $\rho(x) = 1$ for large $|x|$, $\max \rho > 1$. Then the perturbed wave guide

$$\Omega := \cap_{x\in\mathbb{R}^k} (\{x\} \times \rho(x) D)$$

has an unbounded sequence of multiple eigenvalues embedded into the continuous spectrum. Notice that the presence of embedded eigenvalues and hence of stationary solutions is in contrast with the decay of the solution. Thus we see that suitable conditions of *repuslivity* on the shape of the domain are essential in order to exclude eigenvalues and ensure dispersion; conversely, in presence of bumps in the wrong direction, even small, we expect in general concentration of energy and disruption of dispersion.

In order to ensure dispersion, it is reasonable to assume that the sections of Ω at fixed y

$$\{x \in \mathbb{R}^n : (x, y) \in \Omega\}$$

be nontrapping exterior domains. Actually, in order to prove smoothing we shall need the following stronger condition (see Figure 2):

Definition 1.3. Let Ω be an open subset of $\mathbb{R}^n_x \times \mathbb{R}^m_y$ with Lipschitz boundary, $n, m \geq 1$. We say that Ω is *repulsive with respect to the x variables* if, denoting by ν the exterior normal to $\partial\Omega$, we have at all points of the boundary

$$\nu \cdot (x, 0) \leq 0. \tag{1.17}$$

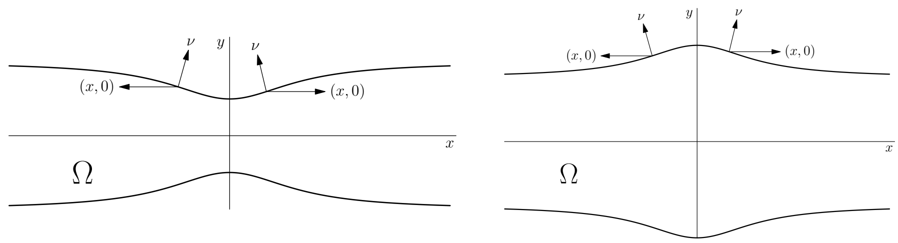

FIGURE 2. A repulsive (left) and nonrepulsive (right) domain w.r.to x

We can now state our results. We shall always consider a waveguide Ω satisfying condition (1.17), with $n \geq 3$ and $m \geq 1$, and a selfadjoint Schrödinger operator

$$H = -\Delta u + V(x, y)$$

with Dirichlet b.c., with a locally bounded potential $V(x, y)$ satisfying the assumptions

$$V \geq 0, \qquad -x \cdot \nabla_x(|x|V) \geq 0. \tag{1.18}$$

The conditions on the potential can be substantially relaxed, for instance by admitting a negative part, small in a suitable sense. We did not strive for maximum generality.

Resolvent estimate. Our approach is based on the Kato smoothing theory (see [9], see also [18]). The crucial tool, which can be considered the fundamental result of the paper, is a uniform resolvent estimate for the operator H. To this end we adapt the method of Morawetz multipliers in the version of [1]. Using the non isotropic Morrey-Campanato norms

$$\|f\|_X = \sup_{R>0} R^{-1/2} \|f\|_{L^2(|x| \leq R)}, \qquad \|f\|_{X_1} = \sup_{R>0} R^{-3/2} \|f\|_{L^2(|x| \leq R)},$$

$$\|f\|_{X^*} = \sum_{j \in \mathbb{Z}} 2^{j/2} \|f\|_{L^2(2^{j-1} \leq |x| \leq 2^j)}$$

(which are asymmetric in x and y), our estimate for the resolvent operator $R(z) = (H - z)^{-1}$ can be stated as follows

$$\|\nabla_x R(z) f\|^2_X + \|R(z) f\|^2_{X_1} + |z| \|R(z) f\|^2_X \leq 5000 n^2 \|f\|^2_{X^*}$$

for all $z \notin \mathbb{R}$ (see Theorem 2.1).

Smoothing estimates. Using the previous resolvent estimate, an application of Kato's theory of smooth operators allows us to prove the following smoothing estimates for the Schrödinger flow e^{itH}

$$\|\langle x\rangle^{-1/2-\epsilon}|D_x|^{1/2}e^{itH}f\|_{L^2_tL^2(\Omega)}+\|\langle x\rangle^{-1-\epsilon}e^{itH}f\|_{L^2_tL^2(\Omega)}\lesssim\|f\|_{L^2(\Omega)},\tag{1.19}$$

while the nonhomogeneous form of the estimates is

$$\begin{aligned}\left\|\langle x\rangle^{-1/2-\epsilon}\int_0^t\nabla_x e^{i(t-s)H}F(s)ds\right\|_{L^2_tL^2(\Omega)}&+\\+\left\|\langle x\rangle^{-1-\epsilon}\int_0^t e^{i(t-s)H}F(s)ds\right\|_{L^2_tL^2(\Omega)}&\lesssim\|\langle x\rangle^{1+\epsilon}F\|_{L^2_tL^2(\Omega)}\end{aligned}\tag{1.20}$$

(see Theorems 3.2, 3.3, 3.4). On the other hand, for the wave-Klein-Gordon equation we prove the estimate ($\mu\geq 0$)

$$\|\langle x\rangle^{-1/2-\epsilon}e^{it\sqrt{H+\mu^2}}f\|_{L^2_tL^2(\Omega)}\lesssim\|f\|_{L^2(\Omega)}\tag{1.21}$$

and, for the inhomogeneous operator,

$$\left\|\int_0^t\langle x\rangle^{-1/2-\epsilon}e^{i(t-s)\sqrt{H+\mu^2}}F(s)ds\right\|_{L^2_tL^2(\Omega)}\lesssim\|\langle x\rangle^{1/2+\epsilon}F\|_{L^2_tL^2(\Omega)}\tag{1.22}$$

(see Theorem 3.5). Notice that our results are comparable with the flat case outlined in Examples 1.1 and 1.2.

Strichartz estimates. A typical application of the smoothing estimates is to deduce Strichartz estimates. We were only able to prove Strichartz estimates for the Schrödinger flow e^{itH}, under the additional assumption that the waveguide Ω coincides with a flat waveguide outside some bounded region. In this case, we can recover the full set of Strichartz estimates, however with a loss of 1/2 derivatives: indeed, we can prove for all $n\geq 3$ and $m\geq 1$ the endpoint estimate

$$\|e^{itH}f\|_{L^2_tL^2_yL_x^{\frac{2n}{n-2}}}\lesssim(1+\|\langle x\rangle^{1+\epsilon}V\|_{L^2_yL^n_x})\Big(\|f\|_{L^2(\Omega)}+\||D_x|^{1/2}f\|_{L^2(\Omega)}\Big)\tag{1.23}$$

(see Theorem 4.1).

Absence of eigenvalues. As an immediate corollary to the smoothing estimates, we deduce that, under the conditions on the domain and on the Schrödinger operator H given above (i.e., $n\geq 3$, $m\geq 1$, Ω repulsive w.r.to x and V as in (1.18)), there are no eigenvalues of H, since the presence of bound states would contradict the L^2 integrability in time of the solution. This generalizes the known results for the special cases in [6] and [15] described in Remark 1.1.

The natural domain of application of our estimates are problems of local and global existence for nonlinear evolution equations. We prefer not to pursue this line of research here; the applications to nonlinear Schrödinger and wave equations on non flat waveguides will be the object of future works.

2. A resolvent estimate

This section is devoted to a study of the resolvent equation $u=R(\lambda+i\epsilon)f$ or equivalently

$$-\Delta u-(\lambda+i\epsilon)u+V(x,y)u=f.\tag{2.1}$$

We shall follow the classical Morawetz multiplier method [14], in the framework of Morrey-Campanato spaces as introduced in [16], see also [2] and [5]. Here additional difficulties are the presence of a boundary, and the necessity to handle the variables x and y in a different way. Moreover, our estimate (2.16) is

stronger than the results in [2] in that it provides a uniform control of the operator $\langle x\rangle^{-1/2-}|z|^{1/2}R(z)\langle x\rangle^{-1/2-}$ (corresponding to the last term at the l.h.s. of (2.16)); this will allow us to prove a sharp smoothing estimate for the wave equation in Theorem 3.5.

The Morrey-Campanato type norms needed here are the following:

$$\|f\|_X = \sup_{R>0} R^{-1/2}\|f\|_{L^2(|x|\le R)}, \qquad \|f\|_{X_1} = \sup_{R>0} R^{-3/2}\|f\|_{L^2(|x|\le R)}, \tag{2.2}$$

$$\|f\|_{X^*} = \sum_{j\in\mathbb{Z}} 2^{j/2}\|f\|_{L^2(2^{j-1}\le|x|\le 2^j)} \tag{2.3}$$

and

$$\|f\|_{X_2} = \sup_{R>0} R^{-1}\|f\|_{L^2(|x|=R)}. \tag{2.4}$$

Notice that the decomposition involves the variables x only. The X^* norm is actually dual to the X norm, but we shall not need this fact. For functions $f \in L^2_{loc}(\Omega)$ we extend the definition of these norms by restriction, meaning that

$$\|f\|_X = \|Ef\|_X, \qquad Ef = f \text{ on } \Omega, \qquad Ef = 0 \text{ on } \mathbb{R}^n \setminus \Omega,$$

We shall use the following elementary inequalities:

$$\|fg\|_{L^1(\Omega)} \le \|f\|_X\|g\|_{X^*}, \tag{2.5}$$

$$\|fg\|_{L^1(\Omega\cap\{R\le|x|\le 2R\})} \le 4R^2\|f\|_X\|g\|_{X_1} \tag{2.6}$$

and

$$\|fgh\|_{L^1(\Omega)} \le 2\|f\|_{X_1}\|g\|_{X^*}\||x|h\|_{L^\infty} \tag{2.7}$$

which implies in particular

$$\|fg\|_{L^1(\Omega\cap\{|x|\le R\})} \le 2R\|f\|_{X_1}\|g\|_{X^*}, \tag{2.8}$$

Moreover it is easy to see that

$$\|f\|_{X_1} \le \|f\|_{X_2}. \tag{2.9}$$

It will also be useful in the following to compare the above norms with standard weighted L^2 norms, with weights of the form

$$\langle x\rangle_R = (R + |x|^2/R)^{1/2}, \qquad \langle x\rangle = (1+|x|^2)^{1/2}. \tag{2.10}$$

We notice that for all real $s > 0$, and for u defined on Ω (after extending u as zero on $\mathbb{R}^n \times \mathbb{R}^m$ outside Ω for simplicity of notation)

$$\begin{aligned}\int (R+|x|^2/R)^{-s}|u|^2 dxdy &\le R^{-s}\int_{|x|\le R}|u|^2 + R^s\int_{|x|>R}|x|^{-2s}|u|^2 \\ &\le R^{-s}\int_{|x|\le R}|u|^2 + 2^{2s}\sum_{j\ge j_R} R^s 2^{-2js}\int_{C_j}|u|^2\end{aligned}$$

where $j_R = [\log_2 R]$ and $C_j = \{(x,y) : 2^{j-1} \le |x| < 2^j\}$. The second term is bounded by

$$\left(\sup_{\rho>0}\rho^{-s}\int_{|x|<\rho}|u|^2\right)2^{2s}R^s\sum_{j\ge j_R}2^{-js} \le \frac{2^{2s}}{1-2^{-s}}\left(\sup_{\rho>0}\rho^{-s}\int_{|x|<\rho}|u|^2\right)$$

so that we have the inequality

$$\int \langle x\rangle_R^{-2s}|u|^2 dxdy \le \frac{2^{4s}}{2^s-1}\sup_{\rho>0}\frac{1}{\rho^s}\int_{|x|<\rho}|u|^2. \tag{2.11}$$

In particular we have, for any $R > 0$,

$$\|\langle x\rangle_R^{-1} u\|_{L^2(\Omega)} \leq 4\|u\|_X, \qquad \|\langle x\rangle_R^{-3} u\|_{L^2(\Omega)} \leq 10\|u\|_{X_1}. \tag{2.12}$$

By a similar proof we obtain for any $R > 0$

$$\|u\|_{X^*} \leq 16\|\langle x\rangle_R u\|_{L^2(\Omega)}. \tag{2.13}$$

Finally, we notice the following inequality, valid for all $\gamma > 0$ and $\epsilon > 0$:

$$\|\langle x\rangle^{-\frac{\gamma}{2}-\epsilon} u\|_{L^2} \leq C(\gamma,\epsilon) \sup_{R>0} \|\langle x\rangle_R^{-\gamma} u\|_{L^2}. \tag{2.14}$$

which evidently holds also with $L^2(\Omega)$ in place of L^2. To prove it is sufficient to write

$$\int \langle x\rangle^{-\gamma-2\epsilon}|u|^2 \leq \int_{|x|\leq 1}|u|^2 + \sum_{j\geq 0} 2^{-j(\gamma+2\epsilon)} \int_{2^j\leq|x|<2^{j+1}}|u|^2$$
$$\leq (1+2^\gamma)\sum_{j\geq 0} 2^{-2j\epsilon} \sup_{R>0}\frac{1}{R^\gamma}\int_{|x|\leq R}|u|^2$$

and observe that

$$\frac{1}{R^\gamma}\mathbf{1}_{|x|\leq R} \leq 2^\gamma \langle x\rangle_R^{-2\gamma}.$$

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^n_x \times \mathbb{R}^m_y$, $n \geq 3$, $m \geq 1$, be a domain repulsive with respect to the variables x, with Lipschitz boundary. Assume the potential $V(x,y)$ satisfies*

$$V \geq 0, \qquad -\partial_x(|x|V) \geq 0 \tag{2.15}$$

and let $u(x,y) \in H^1_0(\Omega)$ be a solution of equation (2.1). Then the following estimate holds:

$$\|\nabla_x u\|_X^2 + \|u\|_{X_1}^2 + (|\lambda|+|\epsilon|)\|u\|_X^2 \leq 5000n^2\|f\|_{X^*}^2. \tag{2.16}$$

Proof. Consider two real valued functions $\psi(x)$ and $\phi(x)$, *independent of the variable y*, such that

$$\nabla\psi, \Delta\psi, \nabla\Delta\psi, \phi, \nabla\phi \text{ are bounded for } |x| \text{ large.} \tag{2.17}$$

and

$$\nu\cdot\nabla\psi \leq 0 \text{ at } \partial\Omega. \tag{2.18}$$

Notice that for a function $\psi(x)$ depending only on x in a radial way, we have

$$\nu\cdot\nabla\psi = \nu\cdot(x,0)|x|^{-1}\partial_x\psi$$

and recalling Definition 1.3, we see that (2.18) is equivalent to the condition that the radial derivative of ψ be non negative:

$$x\cdot\nabla_x\psi \geq 0. \tag{2.19}$$

Then we can form the Morawetz multiplier

$$(\Delta\psi-\phi)\overline{u} + 2\nabla\psi\cdot\nabla\overline{u}. \tag{2.20}$$

Multiplying the resolvent equation (2.1) by the quantity (2.20) and taking the real part we obtain the identity

(2.21)

$$\nabla u(2D^2\psi-\phi I)\nabla\overline{u} + \frac{1}{2}\Delta(\phi-\Delta\psi)|u|^2 + \phi\lambda|u|^2 - (\nabla V\cdot\nabla\psi+\phi V)|u|^2 + \nabla\cdot\Re Q_1 =$$
$$= \nabla\cdot\Re Q + \Re f(2\nabla\psi\cdot\nabla\overline{u} + (\Delta\psi-\phi)\overline{u}) - 2\epsilon\Im(\nabla\psi\cdot\nabla\overline{u}\ u)$$

where

$$Q = \Delta\psi\overline{u}\nabla u - \frac{1}{2}\nabla\Delta\psi|u|^2 - (V-\lambda)\nabla\psi|u|^2 + \frac{1}{2}\nabla\phi|u|^2 - \phi\overline{u}\nabla u \tag{2.22}$$

and

$$Q_1 = \nabla\psi|\nabla u|^2 - 2\nabla u(\nabla\psi \cdot \nabla\overline{u}) \tag{2.23}$$

Our goal is to integrate (2.29) on Ω, with a suitable choice of the weights ϕ and ψ. First of all we show how to handle the last term at the right hand side. Multiplying (2.1) by $\overline{u}$ and splitting real and imaginary parts we obtain the two identities

$$\Im\nabla\cdot\{\nabla u\overline{u}\} + \epsilon|u|^2 = -\Im(f\overline{u}) \qquad \Longrightarrow \qquad \pm\Im\nabla\cdot\{\nabla u\overline{u}\} + |\epsilon||u|^2 = \mp\Im(f\overline{u}) \tag{2.24}$$

$\pm$ being the sign of ϵ, and

$$\Re\nabla\cdot\{-\nabla u\overline{u}\} + |\nabla u|^2 = (\lambda - V)|u|^2 + \Re(f\overline{u}). \tag{2.25}$$

From the second one we deduce (with $\lambda^+ = \max\{\lambda, 0\}$)

$$|\epsilon||\nabla u|^2 \leq |\epsilon|\lambda^+|u|^2 + |\epsilon|\Re(f\overline{u}) + \Re\nabla\cdot\{|\epsilon|\nabla u\overline{u}\}$$

by the positivity of $V(x, y)$, and using (2.24)

$$= \mp\lambda^+\Im(f\overline{u}) + \nabla\cdot\left\{\pm\Im\lambda^+ u\nabla\overline{u} + \Re|\epsilon|\nabla u\overline{u}\right\} + |\epsilon|\Re(f\overline{u})$$

and hence

$$|\epsilon||\nabla u|^2 \leq (\lambda^+ + |\epsilon|)|f\overline{u}| + \nabla\cdot\left\{\pm\Im\lambda^+ u\nabla\overline{u} + \Re|\epsilon|\nabla u\overline{u}\right\}. \tag{2.26}$$

Now by Cauchy-Schwarz we can write

$$2|\epsilon u\nabla\overline{u}| \leq |\epsilon|(\lambda^+ + |\epsilon|)^{1/2}|u|^2 + |\epsilon|(\lambda^+ + |\epsilon|)^{-1/2}|\nabla u|^2$$

and using (2.24), (2.26)

$$\leq 2(\lambda^+ + |\epsilon|)^{1/2}|f\overline{u}| \mp \nabla\cdot\{\Im\nabla\overline{u}u\}(\lambda^+ + |\epsilon|)^{1/2} + \nabla\cdot\left\{\pm\Im\lambda^+ u\nabla\overline{u} + \Re|\epsilon|\nabla u\overline{u}\right\}(\lambda^+ + |\epsilon|)^{-1/2}.$$

In conclusion we have the estimate

$$2|\epsilon u\nabla\overline{u}| \leq 2(\sqrt{|\epsilon|} + \sqrt{\lambda^+})|f\overline{u}| + \nabla\cdot A \tag{2.27}$$

with

$$A = \frac{|\epsilon|\Re\nabla u\overline{u} \pm (2\lambda^+ + |\epsilon|)\Im\nabla\overline{u}u}{(\lambda^+ + |\epsilon|)^{1/2}}, \qquad \pm = \text{ sign of } \epsilon. \tag{2.28}$$

We insert this in our basic identity (2.21) obtaining the inequality

(2.29)

$$\nabla u(2D^2\psi - \phi I)\nabla\overline{u} + \frac{1}{2}\Delta(\phi - \Delta\psi)|u|^2 + \phi\lambda|u|^2 - (\nabla V\cdot\nabla\psi + \phi V)|u|^2 + \nabla\cdot\Re Q_1 \leq$$
$$\leq 2|f\nabla\psi\cdot\nabla\overline{u}| + |f(\Delta\psi - \phi)\overline{u}| + 2\|\nabla\psi\|_{L^\infty}(\sqrt{|\epsilon|} + \sqrt{\lambda^+})|f\overline{u}| + \nabla\cdot\Re P$$

where

$$P = Q + \|\nabla\psi\|_{L^\infty}A \tag{2.30}$$

with A, Q, Q_1 given by (2.28), (2.22) (2.23) respectively.

Next we show how to estimate the integral over Ω of the right hand side of (2.29). We need an additional estimate, obtained by multiplying (2.1) by $\chi\overline{u}$ and taking the imaginary part: as in (2.24) we get

$$\pm\Im\nabla\cdot\{\chi\nabla u\overline{u}\} + |\epsilon|\chi|u|^2 = \mp\Im(\chi f\overline{u}) \mp \Im(\nabla\chi\cdot\nabla\overline{u}u). \tag{2.31}$$

We choose χ as a radial function of the variables x only, and precisely

$$\chi = \begin{cases} 1 & \text{if } |x| < R, \\ 0 & \text{if } |x| > 2R, \\ 2 - |x|/R & \text{if } R \leq |x| \leq 2R. \end{cases}$$

Then integrating (2.31) on Ω and noticing that the boundary terms disappear (thanks to the Dirichlet b.c.), we arrive at the inequality

$$|\epsilon| \int_{\Omega\cap\{|x|\leq R\}} |u|^2 \leq \int_{\Omega\cap\{|x|\leq 2R\}} |f\overline{u}| + \frac{1}{R}\int_{\Omega\cap\{R\leq|x|\leq 2R\}} |\nabla_x u||u|$$

since χ depends only on x. We estimate the right hand side using (2.8), (2.6), and dividing by R we obtain

$$\frac{|\epsilon|}{R}\int_{\Omega\cap\{|x|\leq R\}} |u|^2 \leq 4\|f\|_{X^*}\|u\|_{X_1} + 4\|\nabla_x u\|_X\|u\|_{X_1}$$

and taking the sup in R we conclude

$$|\epsilon|\|u\|_X^2 \leq 4\left(\|f\|_{X^*} + \|\nabla_x u\|_X\right)\|u\|_{X_1} \tag{2.32}$$

Now consider the quantity

$$2(\sqrt{\lambda^+} + \sqrt{|\epsilon|})\|f\overline{u}\|_{L^1(\Omega)} \leq 2(\sqrt{\lambda^+} + \sqrt{|\epsilon|})\|f\|_{X^*}\|u\|_X$$

where we used again (2.5). By (2.32) we have

$$\leq 2\sqrt{\lambda^+}\|f\|_{X^*}\|u\|_X + 4\|f\|_{X^*}(\|f\|_{X^*} + \|\nabla_x u\|_X)^{1/2}\|u\|_{X_1}^{1/2}$$

and hence, for all $\delta \in (0,1)$,

$$2(\sqrt{\lambda^+} + \sqrt{|\epsilon|})\|f\overline{u}\|_{L^1(\Omega)} \leq \delta(\lambda^+\|u\|_X^2 + \|\nabla_x u\|_X^2 + \|u\|_{X_1}^2) + 5\delta^{-1}\|f\|_{X^*}^2. \tag{2.33}$$

This inequality will be used to estimate the third term in the r.h.s. of (2.29).

We consider now the term $\nabla\cdot\Re P = \Re\nabla\cdot(Q + \|\nabla\psi\|_{L^\infty}A)$, which vanishes after integration. To see this, we define the cylinder

$$C_R = \{(x,y)\colon |x| < R,\ y\in\mathbb{R}^m\},$$

we integrate $\nabla\cdot P$ on $\Omega\cap C_R$ and let $R\to+\infty$. The boundary of $\Omega\cap C_R$ is the union of the two sets

$$S_1 = \partial\Omega\cap C_R \quad\text{and}\quad S_2 = \partial C_R\cap\Omega = \{(x,y)\in\Omega\colon |x| = R\}$$

and orrespondingly, we get two surface integrals. The integral on S_1 vanishes thanks to the Dirichlet boundary condition, thus we are left with the boundary integral

$$\int_{S_2} \nu\cdot P d\sigma.$$

By the first assumption (2.17) on the weights ϕ,ψ we have evidently

$$\liminf_{R\to+\infty}\int_{S_2}\nu\cdot P d\sigma = 0 \tag{2.34}$$

since the function u is in $H^1(\Omega)$. This proves that

$$\int_\Omega (\nabla\cdot P)dxdy = 0.$$

Concerning the first and the second term at the right hand side of (2.29), we estimate their integrals using (2.5)

$$2\int_\Omega |f\nabla\psi\cdot\nabla\overline{u}| \leq 2\|\nabla\psi\|_{L^\infty}\|f\|_{X^*}\|\nabla_x u\|_X$$

(recall $\psi = \psi(x)$) and (2.7)

$$\int_\Omega |f(\Delta\psi - \phi)\overline{u}| \leq 2\||x|(\Delta\psi-\phi)\|_{L^\infty}\|f\|_{X^*}\|u\|_{X_1}.$$

Summing up, the integral over Ω of the right hand side of (2.29) is bounded by

$$C(\phi,\psi)\delta(\lambda^+\|u\|_X^2 + \|\nabla_x u\|_X^2 + \|u\|_{X_1}^2) + C(\phi,\psi)\delta^{-1}\|f\|_{X^*}^2 \tag{2.35}$$

with

$$C(\phi,\psi)=10\|\nabla\psi\|_{L^\infty}+10\||x|(\Delta\psi-\phi)\|_{L^\infty}. \tag{2.36}$$

Consider now the left hand side of (2.29). The term in divergence form $\nabla\cdot\Re Q_1$, with

$$Q_1=\nabla\psi|\nabla u|^2-2\nabla u(\nabla\psi\cdot\nabla\overline{u})$$

can be handled as above by integrating first on the cylinder C_R and then letting $R\to+\infty$. The integral on S_2 satisfies again (2.34) and vanishes in the limit. As to the integral on $S_1\subseteq\partial\Omega$, we notice that ∇u at $\partial\Omega$ must be normal to the boundary, because of the Dirichlet boundary condition; in other words, denoting the normal derivative at $\partial\Omega$ with $\partial_\nu u=\nu\cdot\nabla u$, we must have

$$\nabla u=\nu\partial_\nu u\quad\text{at}\quad\partial\Omega$$

so that

$$\nu\cdot\nabla Q_1=\nu\cdot\nabla\psi|\partial_\nu u|^2-2\partial_\nu u(\nabla\psi\cdot\nu\;\partial_\nu\overline{u})=-(\nu\cdot\nabla\psi)|\partial_\nu u|^2.$$

Thus the integral on S_1 can be written

$$I_R=-\int_{S_1}\nu\cdot\nabla\psi|\partial_\nu u|^2d\sigma$$

and under the second assumption (2.18) on the weight ψ we obtain

$$I_R\ge 0\text{ for all }R.$$

Hence we can drop I_R from the computation, and recalling also (2.35) we obtain the basic integral inequality

(2.37)
$$\int_\Omega\left[\nabla u(2D^2\psi-\phi I)\nabla\overline{u}+\frac{1}{2}\Delta(\phi-\Delta\psi)|u|^2+\phi\lambda|u|^2-(\nabla V\cdot\nabla\psi+\phi V)|u|^2\right]\le$$
$$\le C(\phi,\psi)\delta(\lambda^+\|u\|_X^2+\|\nabla_x u\|_X^2+\|u\|_{X_1}^2)+C(\phi,\psi)\delta^{-1}\|f\|_{X^*}^2$$

It remains to choose the functions ϕ,ψ in an appropriate way. When $\lambda>0$ we make the following choice, inspired by [2]:

$$\psi(x,y)=\begin{cases}|x| & \text{if } |x|\ge R,\\ \frac{R}{2}+\frac{|x|^2}{2R} & \text{if } |x|<R,\end{cases}\qquad \phi(x,y)=\begin{cases}0 & \text{if } |x|\ge R,\\ \frac{1}{R} & \text{if } |x|<R.\end{cases} \tag{2.38}$$

Notice that assumptions (2.17) and (2.18) (i.e. (2.19)) are satisfied. We compute the quantities relevant to our estimate: we have

$$\phi-\Delta\psi=\begin{cases}-\frac{n-1}{|x|} & \text{if } |x|\ge R,\\ -\frac{n-1}{R} & \text{if } |x|<R\end{cases}$$

(with a cancelation of the singularity at $|x|=R$). Thus we have, in distribution sense,

$$\Delta(\phi-\Delta\psi)=\frac{n-1}{R^2}\delta_{|x|=R}+\begin{cases}\frac{\mu_n}{|x|^3} & \text{if } |x|\ge R,\\ 0 & \text{if } |x|<R,\end{cases}\qquad \mu_n=(n-1)(n-3)$$

and also

$$\|\nabla\psi\|_{L^\infty}=1,\qquad \||x|(\Delta\psi-\phi)\|_{L^\infty}=n-1\qquad\Longrightarrow\qquad C(\phi,\psi)=10n.$$

For the first term in (2.37) we need the elementary formula, valid for a radial function $\psi=\sigma(|x|)$

$$\nabla uD^2\psi\nabla\overline{u}=\sigma''|\partial_x u|^2+\frac{\sigma'}{|x|}|\nabla_x u-\widehat{x}\;\partial_x u|^2$$

which implies

$$\nabla u(2D^2\psi - \phi I)\nabla \overline{u} = \begin{cases} \frac{2}{R}|\nabla_x u - \widehat{x}\partial_x u|^2 & \text{if } |x| \geq R, \\ \frac{1}{R}|\nabla_x u|^2 & \text{if } |x| < R. \end{cases}$$

Finally, the terms containing the potential V are easily seen to be positive, thanks to assumption (2.15), and we can drop them. Thus (2.37) implies

$$\frac{1}{R}\|\nabla_x u\|^2_{L^2(\Omega\cap\{|x|\leq R\})} + \frac{n-1}{2R^2}\int_{\Omega\cap\{|x|=R\}}|u|^2 d\sigma + \frac{\lambda}{R}\|u\|^2_{L^2(\Omega\cap\{|x|\leq R\})} \leq$$
$$\leq 10n\delta(\lambda\|u\|^2_X + \|\nabla_x u\|^2_X + \|u\|^2_{X_1}) + 10n\delta^{-1}\|f\|^2_{X^*}$$

and taking the sup in $R > 0$ we obtain

$$\|\nabla_x u\|^2_X + \frac{n-1}{2}\|u\|^2_{X_2} + \lambda\|u\|^2_X \leq 10n\delta(\lambda\|u\|^2_X + \|\nabla_x u\|^2_X + \|u\|^2_{X_1}) + 10n\delta^{-1}\|f\|^2_{X^*}$$

Recalling that the X_2 norm dominates the X_1 norm and choosing $\delta = (20n)^{-1}$ we finally obtain in the case $\lambda > 0$

$$\|\nabla_x u\|^2_X + \|u\|^2_{X_1} + \lambda\|u\|^2_X \leq 400n^2\|f\|^2_{X^*}, \qquad \lambda > 0. \tag{2.39}$$

In the case $\lambda \leq 0$ we make a different choice of weights. Following [5], we take simply $\phi \equiv 0$ and we define

$$\psi(x) = \int_0^{|x|}\alpha(r)dr, \qquad \alpha(r) = \begin{cases} \frac{1}{n} - \frac{1}{2n(n+2)}\frac{R^{n-1}}{r^{n-1}} & \text{if } r \geq R, \\ \frac{1}{2n} + \frac{r}{2nR} - \frac{1}{2n(n+2)}\frac{r^3}{R^3} & \text{if } r < R. \end{cases} \tag{2.40}$$

We have now, after some elementary computations,

$$\Delta\psi = \begin{cases} \frac{3(n-1)}{n}\frac{1}{r} & \text{if } r \geq R, \\ \frac{1}{2R} + \frac{n-1}{nr} - \frac{r^2}{2nR^3} & \text{if } r < R, \end{cases} \tag{2.41}$$

moreover

$$\|\nabla\psi\|_{L^\infty} = \frac{1}{n}, \qquad \||x|\Delta\psi\|_{L^\infty} \leq 1 - \frac{1}{n} \qquad \Longrightarrow \qquad C(\phi,\psi) \leq 10,$$

for $n = 3$

$$-\Delta^2\psi = \frac{1}{R^3}\chi_{|x|<R} + 8\pi\delta_0(x)$$

where $\delta_0(x)$ is the Dirac delta at 0 in the variables x and χ_A is the characteristic function of the set A, while for $n \geq 4$ we have ($\mu_n = (n-1)(n-3)$)

$$-\Delta^2\psi = \left(\frac{1}{R^3} + \frac{\mu_n}{2n|x|^3}\right)\chi_{|x|<R} + \frac{\mu_n}{n|x|^3}\chi_{|x|\geq R} + \frac{n-3}{2nR^2}\delta_{|x|=R}$$

so that in all cases $n \geq 3$ we have

$$-\Delta^2\psi \geq \frac{1}{R^3}\chi_{|x|<R}.$$

Moreover,

$$\nabla u D^2\psi\nabla\overline{u} \geq \frac{n-1}{2n(n+2)}\frac{1}{R}|\nabla_x u|^2\chi_{|x|<R}.$$

Thus, proceeding exactly as above, we obtain

$$\frac{n-1}{n(n+2)}\|\nabla_x u\|^2_X + \|u\|^2_{X_1} \leq 10\delta(\|\nabla u\|^2_X + \|u\|^2_{X_1}) + 10\delta^{-1}\|f\|^2_{X^*}$$

and choosing $\delta = (40n)^{-1}$ we conclude, for $\lambda \leq 0$,

$$\|\nabla_x u\|^2_X + \|u\|^2_{X_1} \leq 800n^2\|f\|^2_{X^*}. \tag{2.42}$$

We collect (2.39) and (2.42) in the estimate, valid for all $\lambda \in \mathbb{R}$,

$$\|\nabla_x u\|^2_X + \|u\|^2_{X_1} + \lambda^+\|u\|^2_X \leq 800n^2\|f\|^2_{X^*}. \tag{2.43}$$

As a last step, we show that the factor λ^+ in (2.43) can be improved to $|\lambda|+|\epsilon|$. First of all, recalling (2.32), and using (2.43), we see that

$$|\epsilon|\|u\|_X^2 \le 4(\|f\|_{X^*}^2+\|\nabla u\|_X)\|u\|_{X_1} \le 3320n^2\|f\|_{X^*}^2. \tag{2.44}$$

Assume now $\lambda=-\lambda^-\le 0$. We multiply the resolvent equation (2.1) by $\overline{u}$ and take real parts, obtaining

$$|\nabla u|^2+\lambda^-|u|^2+V|u|^2=\Re(f\overline{u})+\frac{1}{2}\Delta|u|^2;$$

then we multiply by a weight function $\mu(x)$ and we get

$$\mu|\nabla u|^2+(\lambda^-+V)\mu|u|^2=\Re(\mu f\overline{u})+\frac{1}{2}\Delta\mu|u|^2+\nabla\cdot(2^{-1}\nabla(\mu|u|^2)).$$

We now integrate on Ω as above; the term in divergence form vanishes by the Dirichlet b.c., and we obtain, using the positivity of V,

$$\int_\Omega \mu(|\nabla u|^2+\lambda^-|u|^2)\le\int_\Omega \mu|f\overline{u}|+\frac{1}{2}\int_\Omega \Delta\mu|u|^2. \tag{2.45}$$

We now choose $\mu=\Delta\psi$ with ψ defined as in (2.40). Notice that $\Delta\mu=\Delta^2\psi\le 0$ so we can drop the last term from the computation; on the other hand

$$\Delta\psi\ge\frac{1}{2R}\chi_{|x|<R},\qquad \||x|\Delta\psi\|_{L^\infty}\le 1$$

and recalling property (2.7) we obtain

$$\frac{1}{2R}\int_{|x|<R}(|\nabla u|^2+\lambda^-|u|^2)\le 2\|f\|_{X^*}\|u\|_{X_1}.$$

Taking the sup in $R>0$ this gives

$$\|\nabla u\|_X^2+\lambda^-\|u\|_X^2\le 4\|f\|_{X^*}\|u\|_{X_1}\le 120n\|f\|_{X^*}^2 \tag{2.46}$$

again by (2.43).

Collecting (2.46), (2.44) and (2.43) we conclude the proof of (2.16).

□

Remark 2.1. When $z=\lambda+i\epsilon$ does not belong to the spectrum of the selfadjoint operator $H=-\Delta+V$ with Dirichlet b.c. on $L^2(\Omega)$ (this includes some cases when $\epsilon=0$), given an $f\in L^2(\Omega)$, we can represent the solution of (2.1) as $u=R(z)f$, where $R(z)=(H-z)^{-1}$. Since we know that $u\in H_0^1(\Omega)$, all the preceding computations apply and in particular estimate (2.16) holds. As a consequence, using (2.12) and (2.13), we can write for all $R,S>0$,

$$\|\langle x\rangle_R^{-3}R(z)f\|_{L^2(\Omega)}\le 2^{13}n\|\langle x\rangle_S f\|_{L^2(\Omega)}. \tag{2.47}$$

Thus (2.47) is in fact a weighted L^2 estimate for the resolvent $R(z)$. By duality we have the equivalent estimate

$$\|\langle x\rangle_R^{-1}R(z)f\|_{L^2(\Omega)}\le c_n\|\langle x\rangle_S^3 f\|_{L^2(\Omega)} \tag{2.48}$$

and by (complex) interpolation we have also

$$\|\langle x\rangle_R^{-2}R(z)f\|_{L^2(\Omega)}\le c_n\|\langle x\rangle_S^2 f\|_{L^2(\Omega)},$$

uniformly in $z\notin\sigma(H)$, which we shall write more symmetrically as follows:

$$\|\langle x\rangle_R^{-2}R(z)\langle x\rangle_S^{-2}f\|_{L^2(\Omega)}\le c_n\|f\|_{L^2(\Omega)}. \tag{2.49}$$

A similar computation, using the other two terms in (2.16), shows that

$$\|\langle x\rangle_R^{-1}\nabla_x R(z)\langle x\rangle_S^{-1}f\|_{L^2(\Omega)}+|z|^{1/2}\|\langle x\rangle_R^{-1}R(z)\langle x\rangle_S^{-1}f\|_{L^2(\Omega)}\le c_n\|f\|_{L^2(\Omega)} \tag{2.50}$$

uniformly in $z\notin\sigma(H)$. In particular this applies to $z=-\delta$ for all $\delta>0$ since the operator H is positive.

At this point we need the following elementary

Lemma 2.2. *If a linear operator A satisfies for all $R, S > 0$ the estimate*

$$\|\langle x\rangle_R^{-\gamma} A \langle x\rangle_S^{-\gamma} u\|_{L^2} \leq C_0 \|u\|_{L^2} \tag{2.51}$$

with a constant independent of R, S, then it satisfies also, for all $\epsilon > 0$, the estimate

$$\|\langle x\rangle^{-\frac{\gamma}{2}-\epsilon} A \langle x\rangle^{-\frac{\gamma}{2}-\epsilon} u\|_{L^2} \leq C_0 C(\gamma,\epsilon)\|u\|_{L^2}. \tag{2.52}$$

Proof. Write (2.51) in the form

$$\|\langle x\rangle_R^{-\gamma} A v\|_{L^2} \leq C_0 \|\langle x\rangle_S^{\gamma} v\|_{L^2}, \tag{2.53}$$

decompose $v = v_0 + \sum_{j\geq 1} v_j$, with v_j supported in $2^{j-1} \leq |x| < 2^j$ for $j \geq 1$ and v_0 in $|x| < 1$, apply the (2.53) to each v_j with $S = 2^j$, and sum over j (all norms in the rest of the proof are in L^2):

$$\|\langle x\rangle_R^{-\gamma} A v\| \leq \|\langle x\rangle_R^{-\gamma} A v_0\| + \sum_j \|\langle x\rangle_R^{-\gamma} A v_j\| \leq C_0 \|\langle x\rangle^{\gamma} v_0\| + C_0 \sum \|\langle x\rangle_{2^j}^{\gamma} v_j\|.$$

Now notice that for $j \geq 1$

$$\langle x\rangle_{2^j}^{2\gamma} = \left(2^j + \frac{|x|^2}{2^j}\right)^{\gamma} \leq 2^{\gamma} 2^{\gamma j} \leq 2^{2\gamma}|x|^{\gamma} \leq 2^{2(\gamma+\epsilon)} 2^{-2\epsilon j}|x|^{\gamma+2\epsilon} \quad \text{on the support of } v_j$$

so that

$$\|\langle x\rangle_R^{-\gamma} A v\| \leq C_0 2^{\gamma}\|v_0\| + C_0 2^{\gamma+\epsilon} \sum_{j\geq 1} 2^{-\epsilon j} \||x|^{\frac{\gamma}{2}+\epsilon} v_j\| \leq C_0 C(\gamma,\epsilon) \|\langle x\rangle^{\frac{\gamma}{2}+\epsilon} v\|$$

by Cauchy-Schwarz. Using (2.14) we obtain (2.52). □

In particular, applying the Lemma to (2.49) and to (2.50) we obtain the estimates, valid for all $\epsilon > 0$:

$$\|\langle x\rangle^{-1-\epsilon} R(z) \langle x\rangle^{-1-\epsilon} f\|_{L^2(\Omega)} \leq c_{n,\epsilon}\|f\|_{L^2(\Omega)}, \tag{2.54}$$

$$\|\langle x\rangle^{-\frac{1}{2}-\epsilon} \nabla_x R(z) \langle x\rangle^{-\frac{1}{2}-\epsilon} f\|_{L^2(\Omega)} \leq c_{n,\epsilon}\|f\|_{L^2(\Omega)}, \tag{2.55}$$

$$|z|^{1/2} \|\langle x\rangle^{-\frac{1}{2}-\epsilon} R(z) \langle x\rangle^{-\frac{1}{2}-\epsilon} f\|_{L^2(\Omega)} \leq c_{n,\epsilon}\|f\|_{L^2(\Omega)}. \tag{2.56}$$

3. Smoothing estimates

The concept of *H-smoothing* was introduced by Kato [9] in the context of scattering theory, and its usefulness for dispersive equations was revealed in [18]. An operator A is *H-smooth* (actually, supersmooth) whenever one of the two equivalent estimates (3.1), (3.2) in the following theorem holds. We shall use a version of the result adapted to the applications we have in mind; for a more complete reference see [17], [**?**]

Theorem 3.1 (Kato). *Assume K is a selfadjoint operator in a Hilbert space $\mathcal{H}$, let $\mathcal{R}(z) = (K - z)^{-1}$ be its resolvent operator for $z \in \mathbb{C} \setminus \mathbb{R}$, and let A be a densely defined closed operator from $\mathcal{H}$ to a second Hilbert space $\mathcal{H}_1$ with $D(A) \supseteq D(K)$.*

Assume that $A, \mathcal{R}(z)$ satisfy the estimate

$$\sup_{z \notin \mathbb{R}} \|A\mathcal{R}(z)A^* f\|_{\mathcal{H}_1} \leq c_0^2 \|f\|_{\mathcal{H}_1} \tag{3.1}$$

for all $f \in D(A^*)$. Then the following estimates hold:

$$\|Ae^{itK} f\|_{L_t^2 \mathcal{H}_1} \leq c_0 \|f\|_{\mathcal{H}}, \tag{3.2}$$

$$\left\| \int_0^t A e^{i(t-s)K} A^* h(s) ds \right\|_{L_t^2 \mathcal{H}_1} \leq c_0^2 \|h\|_{L_t^2 \mathcal{H}_1} \tag{3.3}$$

for all $f \in \mathcal{H}$, $h \in L^2_t\mathcal{H}_1$.

Estimate (3.2) still holds when (3.1) is replaced by the weaker assumption

$$\sup_{z\notin\mathbb{R}} \|A\Im(\mathcal{R}(z))A^*f\|_{\mathcal{H}_1} \le c_0^2\|f\|_{\mathcal{H}_1}, \tag{3.4}$$

where we use the notation $\Im T = (2i)^{-1}(T - T^*)$.

Recalling (2.49) in Remark 2.1, we see that with the choices

$$\mathcal{H} = \mathcal{H}_1 = L^2(\Omega), \qquad K = H = -\Delta + V(x,y), \qquad A = \langle x\rangle^{-1-\epsilon}$$

estimate (2.54) reduces precisely to (3.1). Thus from Theorem 3.1 and (2.54) we obtain immediately the following smoothing estimates for the Schrödinger flow associated to the operator $H = -\Delta + V(x,y)$:

Theorem 3.2. *Let the domain $\Omega \subseteq \mathbb{R}^n_x \times \mathbb{R}^m_y$, $n \ge 3$, $m \ge 1$ be repulsive with respect to the x variables, with a Lipschitz boundary. Assume the operator $H = -\Delta + V(x,y)$ with Dirichlet b.c. is selfadjoint on $L^2(\Omega)$. Finally, assume that the potential V satisfies on Ω the inequalities*

$$V(x,y) \ge 0, \qquad -\partial_x(|x|V(x,y)) \ge 0. \tag{3.5}$$

Then the Schrödinger flow associated to H satisfies the following smoothing estimates: for any $\epsilon > 0$,

$$\|\langle x\rangle^{-1-\epsilon}e^{itH}f\|_{L^2_tL^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \tag{3.6}$$

$$\left\|\langle x\rangle^{-1-\epsilon}\int_0^t e^{i(t-s)H}F(s)ds\right\|_{L^2_tL^2(\Omega)} \lesssim \|\langle x\rangle^{1+\epsilon}F\|_{L^2_tL^2(\Omega)} \tag{3.7}$$

for all $f(x,y) \in L^2(\Omega)$ and $F(t,x,y)$ with $\langle x\rangle^{1+\epsilon}F \in L^2_tL^2(\Omega)$.

We can obtain an estimate also for the derivatives of $e^{itH}f$, with a gain of a half derivative, by a different choice of the operator A and some functional analytic arguments; to this end we must introduce suitable functional spaces.

For functions on $\mathbb{R}^{n+m}$ and $z \in \mathbb{C}$, we introduce the operators acting only on the x variables

$$|D_x|^z f(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} |\xi|^z\widehat{f}(\xi,y)e^{i\xi x}dx,$$
$$\langle D_x\rangle^z f(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} \langle\xi\rangle^z\widehat{f}(\xi,y)e^{i\xi x}dx,$$

where $\widehat{f}(\xi,y)$ is the Fourier transform of $f(x,y)$ with respect to the variable x only. By standard calculus we have the equivalence

$$\||D_x|f\|_{L^2(\mathbb{R}^{n+m})} \simeq \|\nabla_x f\|_{L^2(\mathbb{R}^{n+m})}.$$

We introduce the norms, and the corresponding Hilbert spaces,

$$\|f\|_{\dot{H}^{s,0}} = \||D_x|^s f\|_{L^2(\mathbb{R}^{n+m})}, \qquad \|f\|_{H^{s,0}} = \|\langle D_x\rangle^s f\|_{L^2(\mathbb{R}^{n+m})}. \tag{3.8}$$

Notice that, if the boundary of Ω satisfies a uniform Lipschitz condition, the extension as 0 of a function $f \in H^1_0(\Omega)$ to all of $\mathbb{R}^{n+m}$ gives a function $Ef \in H^1\mathbb{R}^{n+m}$ with the same norm. Thus for $f \in H^1_0(\Omega)$ and $0 \le \Re z \le 1$ we can extend the definition of the operators as

$$|D_x|^z f = |D_x|^z Ef, \qquad \langle D_x\rangle^z f = |D_x|^z Ef.$$

By density of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$ we obtain also that

$$\||D_x|f\|_{L^2(\Omega)} \simeq \|\nabla_x f\|_{L^2(\Omega))}. \tag{3.9}$$

Recall now the estimate ($y \in \mathbb{R}$)

$$\|\langle x\rangle^{-s}|D_x|^{1+iy}f\|_{L^2(\mathbb{R}^{n+m})} \simeq \|\langle x\rangle^{-s}\nabla_x f\|_{L^2(\mathbb{R}^{n+m})}, \qquad s > -\frac{n}{2} \tag{3.10}$$

which holds since the Riesz operators $\partial_{x_j}|D_x|^{-1}$ and the operators $|D_x|^{iy}$ are bounded in weighted L^2 with A_2 weights, and $\langle x\rangle^{-s} \in A_2(\mathbb{R}^n)$ for $s > -n/2$; notice also that the constant in the estimate depends on $y \in \mathbb{R}$ but with a polynomial growth as $|y| \to \infty$ (see [20] for the general theory of singular integrals in weighted L^2 spaces, and more specifically [19], [4] for the polynomial growth of the norms). The estimate extends to

$$\|\langle x\rangle^{-s}|D_x|^{1+iy}f\|_{L^2(\Omega)} \simeq \|\langle x\rangle^{-s}\nabla_x f\|_{L^2(\Omega))}, \qquad s > -\frac{n}{2}, \quad f \in H^1_0(\Omega) \tag{3.11}$$

by a density argument as above.

As a consequence of (3.11), (2.55) implies the estimate

$$\|\langle x\rangle^{-1/2-\epsilon}|D_x|^{1+iy}R(z)\langle x\rangle^{-1/2-\epsilon}f\|_{L^2(\Omega)} \leq c_{n,\epsilon}\|f\|_{L^2(\Omega)} \tag{3.12}$$

which by duality is equivalent to

$$\|\langle x\rangle^{-1/2-\epsilon}R(z)|D_x|^{1+iy}\langle x\rangle^{-1/2-\epsilon}f\|_{L^2(\Omega)} \leq c_{n,\epsilon}\|f\|_{L^2(\Omega)} \tag{3.13}$$

Thus by complex interpolation for the analytic family of operators $T_z =$ we also obtain the estimate

$$\|\langle x\rangle^{-1/2-\epsilon}|D_x|^{1/2}R(z)|D_x|^{1/2}\langle x\rangle^{-1/2-\epsilon}f\|_{L^2(\Omega)} \leq c_{n,\epsilon}\|f\|_{L^2(\Omega)} \tag{3.14}$$

Now we make the following choice:

$$\mathcal{H} = \dot{H}^{1/2,0}(\Omega), \qquad \mathcal{H}_1 = L^2(\Omega), \qquad H = -\Delta + V(x,y)$$

where the space $\dot{H}^{1/2,0}(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ in the norm

$$\|f\|_{\dot{H}^{1/2,0}(\Omega)} = \||D_x|^{1/2}f\|_{L^2(\Omega)},$$

The closed unbounded operator $A : \mathcal{H} \to \mathcal{H}_1$ is now defined as

$$A = \langle x\rangle^{-1/2-\epsilon}|D_x|$$

and its adjoint A^* is computed as follows

$$\begin{aligned}(Af, g)_{\mathcal{H}_1} =&(\langle x\rangle^{-1/2-\epsilon}|D_x|f, g)_{L^2(\Omega)} = (|D_x|f, \langle x\rangle^{-1/2-\epsilon}g)_{L^2(\Omega)} = \\ &= (|D_x|^{1/2}f, |D_x|^{1/2}\langle x\rangle^{-1/2-\epsilon}g)_{L^2(\Omega)} = (f, \langle x\rangle^{-1/2-\epsilon}g)_{\mathcal{H}} = (f, A^*g)_{\mathcal{H}}.\end{aligned}$$

With these choices, estimate (2.55) takes precisely the form (3.1) and Kato theory applies. We obtain the following

Theorem 3.3. *Assume Ω, V, H as in Theorem 3.2. Then the Schrödinger flow associated to H satisfies the smoothing estimates*

$$\|\langle x\rangle^{-1/2-\epsilon}\nabla_x e^{itH}f\|_{L^2_tL^2(\Omega)} \lesssim \||D_x|^{1/2}f\|_{L^2(\Omega)}, \tag{3.15}$$

$$\left\|\langle x\rangle^{-1/2-\epsilon}\int_0^t \nabla_x e^{i(t-s)H}F(s)ds\right\|_{L^2_tL^2(\Omega)} \lesssim \|\langle x\rangle^{1/2+\epsilon}F\|_{L^2_tL^2(\Omega)} \tag{3.16}$$

for all $f(x,y) \in H^1_0(\Omega)$ and $F(t,x,y)$ with $\langle x\rangle^{1/2+\epsilon}F \in L^2_tL^2(\Omega)$.

Notice that a different choice is possible: namely, if we set

$$\mathcal{H} = \mathcal{H}_1 = L^2(\Omega), \qquad H = -\Delta + V(x,y)$$

and

$$A = \langle x\rangle^{-1/2-\epsilon}|D_x|^{1/2}, \qquad A^* = |D_x|^{1/2}\langle x\rangle^{-1/2-\epsilon}$$

we obtain the (essentially equivalent) result:

Theorem 3.4. *Assume Ω, V, H as in Theorem 3.2. Then the Schrödinger flow associated to H satisfies the smoothing estimates*

$$\|\langle x\rangle^{-1/2-\epsilon}|D_x|^{1/2}e^{itH}f\|_{L^2_tL^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \tag{3.17}$$

$$\left\|\langle x\rangle^{-1/2-\epsilon}\int_0^t |D_x|^{1/2}e^{i(t-s)H}F(s)ds\right\|_{L^2_tL^2(\Omega)} \lesssim \|\langle x\rangle^{1/2+\epsilon}|D_x|^{-1/2}F\|_{L^2_tL^2(\Omega)} \tag{3.18}$$

for all $f(x,y)\in L^2(\Omega)$ and $F(t,x,y)$ with $\langle x\rangle^{1/2+\epsilon}F\in L^2_tL^2(\Omega)$.

Handling the wave and Klein-Gordon equations requires some additional effort. We start from the standard representation

$$K=\begin{pmatrix}0 & 1\\ H & 0\end{pmatrix} \qquad\Longrightarrow\qquad \exp(itK)=\begin{pmatrix}\cos(t\sqrt{H}) & \frac{i}{\sqrt{H}}\sin(t\sqrt{H})\\ i\sqrt{H}\sin(t\sqrt{H}) & \cos(t\sqrt{H})\end{pmatrix} \tag{3.19}$$

so that

$$e^{itK}\begin{pmatrix}f\\ \sqrt{H}f\end{pmatrix}=\begin{pmatrix}e^{it\sqrt{H}}f\\ \sqrt{H}e^{it\sqrt{H}}f\end{pmatrix} \tag{3.20}$$

is the flow associated to the wave equation

$$u_{tt}+Hu=0.$$

Now we choose

$$\mathcal{H}=D(\sqrt{H})\times L^2(\Omega), \qquad \mathcal{H}_1=L^2(\Omega), \qquad H=-\Delta+V(x,y)$$

with K as in (3.19), and $A:\mathcal{H}\to L^2(\Omega)$ defined by

$$A\begin{pmatrix}f\\ g\end{pmatrix}=\langle x\rangle^{-1/2-\epsilon}H^{1/2}f \qquad\Longrightarrow\qquad A^*g=\begin{pmatrix}H^{-1/2}\langle x\rangle^{-1/2-\epsilon}g\\ 0\end{pmatrix}.$$

Then the resolvent $\mathcal{R}(z)=(K-z)^{-1}$ can be written in terms of the resolvent $R(z)=(H-z)^{-1}$ as

$$\mathcal{R}(z)=\begin{pmatrix}zR(z^2) & R(z^2)\\ HR(z^2) & zR(z^2)\end{pmatrix}. \tag{3.21}$$

Thus we see that, in order to apply the Kato theory to e^{itK}, we need to prove that the following operator is bounded on $L^2(\Omega)$, uniformly in $z\notin\mathbb{R}$:

$$Q(z)=A\mathcal{R}(z)A^*\equiv\langle x\rangle^{-1/2-\epsilon}zR(z^2)\langle x\rangle^{-1/2-\epsilon}.$$

This is precisely what is expressed by estimate (2.56). Then by Theorem 3.1 we obtain

$$\left\|Ae^{itK}\begin{pmatrix}f\\ \sqrt{H}f\end{pmatrix}\right\|_{L^2_t\mathcal{H}_1}\lesssim\left\|\begin{pmatrix}f\\ \sqrt{H}f\end{pmatrix}\right\|_{\mathcal{H}}$$

which means

$$\|\langle x\rangle^{-1/2-\epsilon}H^{1/2}e^{it\sqrt{H}}f\|_{L^2_tL^2(\Omega)}\lesssim\|H^{1/2}f\|_{L^2(\Omega)}$$

or equivalently

$$\|\langle x\rangle^{-1/2-\epsilon}e^{it\sqrt{H}}f\|_{L^2_tL^2(\Omega)}\lesssim\|f\|_{L^2(\Omega)}.$$

A similar estimate is obtained for the Duhamel term. All the previous computations are valid if we replace the operator H with $H+\mu^2$, $\mu\ge 0$; this gives an analogous estimate for the flow $e^{it\sqrt{H+\mu^2}}$ associated to the Klein-Gordon equation. In conclusion, we have proved:

Theorem 3.5. *Let $\mu \geq 0$ and assume Ω, V, H as in Theorem 3.2. Then the wave flow associated to $H + \mu^2$ satisfies the smoothing estimates*

$$\|\langle x\rangle^{-1/2-\epsilon} e^{it\sqrt{H+\mu^2}} f\|_{L^2_t L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \tag{3.22}$$

$$\left\|\langle x\rangle^{-1/2-\epsilon} \int_0^t e^{i(t-s)\sqrt{H+\mu^2}} F(s)ds\right\|_{L^2_t L^2(\Omega)} \lesssim \|\langle x\rangle^{1/2+\epsilon} F\|_{L^2_t L^2(\Omega)} \tag{3.23}$$

for all $f(x,y) \in L^2(\Omega)$ and $F(t,x,y)$ with $\langle x\rangle^{1/2+\epsilon} F \in L^2_t L^2(\Omega)$.

4. Strichartz estimates for the Schrödinger equation

From now on we reduce to the simpler situation when the domain Ω, besides being x-repulsive, is a compactly supported perturbation of a product domain. More precisely we assume that there exist a constant M and an open set $\omega \subseteq \mathbb{R}^m$ such that

$$\Omega \cap \{(x,y) : |x| > M\} = (\mathbb{R}^n \times \omega) \cap \{(x,y) : |x| > M\}. \tag{4.1}$$

We recall the estimates proved in Example 1.1 in the flat case

$$\|e^{it\Delta} f\|_{L^2_t L^2_y L_x^{\frac{2n}{n-2}}} \lesssim \|f\|_{L^2(\Omega)}, \qquad \left\|\int_0^t e^{i(t-s)\Delta} F(s)ds\right\|_{L^2_t L^2_y L_x^{\frac{2n}{n-2}}} \lesssim \|F\|_{L^2_t L^2_y L_x^{\frac{2n}{n+2}}} \tag{4.2}$$

where Δ is the Dirichlet Laplacian on $\mathbb{R}^n \times \omega$. In the following, we shall also need a mixed Strichartz-smoothing nonhomogeneous estimate, which follows like (4.2) from a corresponding estimate on the whole space. Indeed, Ionescu and Kenig proved that for the standard Laplace operator on $\mathbb{R}^n$, $n \geq 3$, one has

$$\left\|\int_0^t e^{i(t-s)\Delta} F(s)ds\right\|_{L^2_t L_x^{\frac{2n}{n+2}}} \lesssim \|\langle x\rangle^{1/2+\epsilon} |D|^{-1/2} F\|_{L^2_t L^2_x} \tag{4.3}$$

(see Lemma 3 in [8], which is actually the dual form of (4.3), and in a sharper version). By mimicking the proof of (4.2) we obtain the following mixed estimate on a flat waveguide:

$$\left\|\int_0^t e^{i(t-s)\Delta} F(s)ds\right\|_{L^2_t L^2_y L_x^{\frac{2n}{n+2}}} \lesssim \|\langle x\rangle^{1/2+\epsilon} |D_x|^{-1/2} F\|_{L^2_t L^2_y L^2_x} \tag{4.4}$$

where again Δ denotes the Dirichlet Laplacian on $\mathbb{R}^n \times \omega$.

Assume now the domain Ω is repulsive with respect to x and satisfies in addition the condition (4.1), and let $u(t,x,y)$ be a solution on Ω of the equation

$$iu_t - \Delta u + V(x,y)u = 0, \qquad u(0,x,y) = f(x,y) \tag{4.5}$$

Recall that by (3.6), (3.15) and (3.17) u satisfies

$$\|\langle x\rangle^{-1-\epsilon} u\|_{L^2_t L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \qquad \|\langle x\rangle^{-1/2-\epsilon} \nabla u\|_{L^2_t L^2(\Omega)} \lesssim \||D_x|^{1/2} f\|_{L^2(\Omega)} \tag{4.6}$$

and

$$\|\langle x\rangle^{-1/2-\epsilon} |D_x|^{1/2} u\|_{L^2_t L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \tag{4.7}$$

Fix a cutoff function $\chi(x)$ equal to 1 on the ball $B(0.M)$ and vanishing outside $B(0, M+1)$ and split the solution as

$$u = v + w, \qquad v(t,x,y) = \chi(x) u(t,x,y), \quad w(t,x,y) = (1-\chi(x)) u(t,x,y).$$

Then w is a solution of the following Schrödinger equation

$$iw_t - \Delta w = G_1 + G_2, \qquad G_1 = -V(x,y)(1-\chi)u + \Delta_x \chi\, u, \quad G_2 = 2\nabla_x \chi \cdot \nabla_x u, \tag{4.8}$$
$$w(0,x,y) = (1-\chi(x)) f(x,y)$$

on $\mathbb{R}^n \times \omega$ with Dirichlet boundary conditions. We can now represent $w(t,x,y)$ as

$$w = e^{it\Delta}(1-\chi)f + i\int_0^t e^{i(t-s)\Delta}G_1(s)ds + i\int_0^t e^{i(t-s)\Delta}G_2(s)ds \equiv I + II + III.$$

We plan to use estimates (4.2) on the first two terms and (4.4) on the third one. The $L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}$ norm of the first term I is estimated directly using (4.2). Again by (4.2), the $L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}$ norm of II is estimated using Hölder's inequality as follows

$$\|\Delta_x \chi(x) u\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n+2}}} \lesssim \|\langle x\rangle^{-1-\epsilon} u\|_{L_t^2 L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \tag{4.9}$$

and

$$\|Vu\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n+2}}} \le \|\langle x\rangle^{1+\epsilon} V\|_{L_y^2 L_x^n} \|\langle x\rangle^{-1-\epsilon} u\|_{L_t^2 L^2(\Omega)} \lesssim \|\langle x\rangle^{1+\epsilon} V\|_{L_y^2 L_x^n} \|f\|_{L^2(\Omega)}. \tag{4.10}$$

using the smoothing estimate (3.6) in both cases. For the third term III, on the other hand, we use the mixed estimate (4.4) so that

$$\|III\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}} \lesssim \|\langle x\rangle^{1/2+\epsilon} |D_x|^{-1/2} (\nabla_x \chi \cdot \nabla_x u)\|_{L_t^2 L_y^2 L_x^2}.$$

Let now $\psi(x)$ be a cutoff function supported in $|x| \le M+3$ and equal to 1 on $|x| \le M+1$ (note χ is supported in $B(0, M+1)$) and recall the explicit formula

$$|D_x|^{-1/2} g = c_n \int \frac{g(z)}{|x-z|^{n-1/2}} dz$$

(here and in the following, integrals extend over all $\mathbb{R}^n$). After integration by parts we can split the quantity to estimate as follows:

$$\langle x\rangle^{1/2+\epsilon} |D_x|^{-1/2} (\nabla_x \chi \cdot \nabla_x u) \simeq \int \beta(x,z) u(z) dz + \int \gamma(x,z) \nabla u(z) dz$$

where

$$\beta(x,z) = -\nabla_z \left(\frac{\langle x\rangle^{1/2+\epsilon} \psi(x) [\nabla\chi(z) - \nabla\chi(x)]}{|x-z|^{n-1/2}} \right)$$

and

$$\gamma(x,z) = \frac{\langle x\rangle^{1/2+\epsilon} \psi(x) \nabla\chi(x)}{|x-z|^{n-1/2}}.$$

In the following we extend the function u as 0 outside Ω but keep the same notation for brevity. We have

$$\int \gamma(x,z) u(z) dz = \langle x\rangle^{1/2+\epsilon} \psi(x) \nabla\chi(x) |D_x|^{-1/2} \nabla_x u$$

which implies, since ψ has compact support,

$$\left\| \int \gamma(x,z) u(z) dz \right\|_{L_x^2} \lesssim \|\langle x\rangle^{-1-\epsilon} |D_x|^{-1/2} \nabla_x u\|_{L_x^2} \lesssim \|\langle x\rangle^{-1-\epsilon} |D_x|^{1/2} u\|_{L_x^2}$$

where in the last step we used (3.11). Finally, β satisfies for all N

$$|\beta(x,z)| \lesssim \langle z\rangle^{-N} |x-z|^{\frac{1}{2}-n}$$

so that

$$\left\| \int \beta(x,z) u(z) dz \right\|_{L_x^2} \lesssim \| |x|^{\frac{1}{2}-n} * (\langle z\rangle^{-N} u) \|_{L_x^2} \lesssim \|\langle z\rangle^{-N} u\|_{L^{\frac{2n}{n+2}}} \lesssim \|\langle x\rangle^{-1-\epsilon} u\|_{L_x^2}$$

by Hardy-Littlewood-Sobolev followed by Hölder's inequality (for N large enough). Summing up, and integrating also in the remaining variables t, y, we arrive at

$$\|III\|_{L_t^2 L_y^2 L_x^{\frac{2n}{n-2}}} \lesssim \|\langle x\rangle^{-1-\epsilon} u\|_{L_t^2 L_y^2 L_x^2(\Omega)} + \|\langle x\rangle^{-1/2-\epsilon} |D_x|^{1/2} u\|_{L_t^2 L_y^2 L_x^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$$

by (4.6), (4.7). In conclusion, putting together the estimates for I, II, III, we obtain

$$\|w\|_{L^2_t L^2_y L_x^{\frac{2n}{n-2}}} \lesssim (1 + \|\langle x\rangle^{1+\epsilon} V\|_{L^2_y L^{\frac{n}{2}}_x}) \|f\|_{L^2(\Omega)}. \tag{4.11}$$

The remaining part $v = \chi(x)u$ can be estimated directly via the Sobolev embedding

$$\|g\|_{L^{\frac{2n}{n-2}}(A)} \lesssim \|\nabla g\|_{L^2(A)} \tag{4.12}$$

which holds for any open set $A \subset \mathbb{R}^n$ (even unbounded) and any $g \in H^1_0(\Omega)$, with a constant independent of A. Then we have

$$\|\chi u\|_{L^2_t L^2_y L_x^{\frac{2n}{n-2}}} \lesssim \|u \nabla \chi\|_{L^2_t L^2_{x,y}} + \|\chi \nabla u\|_{L^2_t L^2_{x,y}} \lesssim \|f\|_{L^2(\Omega)} + \||D_x|^{1/2} f\|_{L^2(\Omega)} \tag{4.13}$$

again by (4.6). Summing up (4.11) and (4.13), we have proved the following

Theorem 4.1. *Assume the domain $\Omega \subseteq \mathbb{R}^n_x \times \mathbb{R}^m_y$, with $n \geq 3$ and $m \geq 1$, has a Lipschitz boundary, is repulsive w.r.to the x variables and satisfies assumption (4.1). Assume the potential $V(x, y)$ satisfies on Ω the inequalities*

$$V(x, y) \geq 0, \qquad -\partial_x(|x| V(x, y)) \geq 0. \tag{4.14}$$

and the operator $H = -\Delta_{x,y} + V(x, y)$ with Dirichlet boundary conditions is self-adjoint on $L^2(\Omega)$. Then the Schrödinger flow of H satisfies the following endpoint Strichartz estimate for all $f \in H^1_0(\Omega)$:

$$\|e^{itH} f\|_{L^2_t L^2_y L_x^{\frac{2n}{n-2}}} \lesssim (1 + \|\langle x\rangle^{1+\epsilon} V\|_{L^2_y L^{\frac{n}{2}}_x}) \Big(\|f\|_{L^2(\Omega)} + \||D_x|^{1/2} f\|_{L^2(\Omega)} \Big). \tag{4.15}$$

Remark 4.1. Proving *nonhomogeneous* Strichartz estimates is more difficult because of analytical technicalities. Recall that the solution to the nonhomogenous Schrödinger equation

$$iu_t + Hu = F(t, x, y), \qquad u(0, x, y) = f(x, y) \tag{4.16}$$

on Ω can be represented as

$$u = e^{itH} f + i \int_0^t e^{i(t-s)H} F(s) ds;$$

we have already estimated the first term in Theorem 4.1, and it remains to study the Duhamel operator

$$\int_0^t e^{i(t-s)H} F(s) ds. \tag{4.17}$$

To this end, we introduce the norm

$$\|g\|_{H^{1/2,0}(\Omega)} = \|g\|_{L^2(\Omega)} + \||D_x|^{1/2} g\|_{L^2(\Omega)} \simeq \|\langle D_x\rangle^{1/2} g\|_{L^2(\Omega)} \tag{4.18}$$

and the corresponding Hilbert space $H^{1/2,0}(\Omega)$ defined as the closure of $C^\infty_c(\Omega)$ in this norm. Moreover we denote by $H^{-1/2,0}(\Omega)$ the dual of this space; its norm can be characterized as

$$\|g\|_{H^{-1/2,0}(\Omega)} \simeq \|\langle D_x\rangle^{-1/2} g\|_{L^2(\Omega)}.$$

Then estimate (4.15) can be written

$$\|e^{itH} f\|_{L^2_t L^2_y L_x^{\frac{2n}{n-2}}} \lesssim \|f\|_{H^{1/2,0}(\Omega)}, \qquad f \in H^1_0(\Omega). \tag{4.19}$$

By interpolation with the conservation of energy

$$\|e^{itH} f\|_{L^2_t L^2(\Omega)} = \|f\|_{L^2(\Omega)} \leq \|f\|_{H^{1/2,0}(\Omega)}$$

we obtain the full family of Strichartz estimates

$$\|e^{itH}f\|_{L^p_t L^2_y L^q_x} \lesssim \|f\|_{H^{1/2,0}(\Omega)}, \tag{4.20}$$

for all *admissible couples* (p,q) of indices, i.e., such that

$$\frac{n}{2} = \frac{2}{p} + \frac{n}{q}, \quad 2 \leq q \leq \frac{2n}{n-2}. \tag{4.21}$$

By duality, for any $F(t,x,y) \in L^2_t H^1_0(\Omega)$, we have also

$$\left\| \langle D_x \rangle^{-1/2} \int e^{-sH} F(s) ds \right\|_{L^2(\Omega)} \leq C(V) \|F\|_{L^{p'}_t L^2_y L^{q'}_x} \tag{4.22}$$

for (p,q) admissible. We also notice that estimates (4.20) can be written in the form

$$\|e^{itH}\langle D_x\rangle^{-1} f\|_{L^p_t L^2_y L^q_x} \lesssim \|\langle D_x\rangle^{-1/2} f\|_{L^2(\Omega)}, \qquad \frac{n}{2} = \frac{2}{p} + \frac{n}{q}, \quad 2 \leq q \leq \frac{2n}{n-2}. \tag{4.23}$$

Now we can combine (4.22) and (4.23) to obtain

$$\left\| \int e^{itH} \langle D_x \rangle^{-1} e^{-isH} F(s) ds \right\|_{L^p_t L^2_y L^q_x} \lesssim C(V) \|F\|_{L^{\widetilde{p}'}_t L^2_y L^{\widetilde{q}'}_x}. \tag{4.24}$$

We can apply a standard trick and use the Christ-Kiselev lemma as in [10], which permits to replace the integral over $\mathbb{R}$ with a truncated integral over $[0,t]$, provided the indices satisfy the additional condition $p > \widetilde{p}'$. This implies the estimate

$$\left\| \int_0^t e^{itH} \langle D_x \rangle^{-1} e^{-isH} F(s) ds \right\|_{L^p_t L^2_y L^q_x} \lesssim C(V) \|F\|_{L^{\widetilde{p}'}_t L^2_y L^{\widetilde{q}'}_x} \tag{4.25}$$

for all (p,q) and $(\widetilde{p},\widetilde{q})$ admissible such that $(p,\widetilde{p}) \neq (2,2)$. To complete the proof we would need an additional functional analytic assumption: *the operator* $\langle D_x \rangle$ *commutes with the flow* e^{itH}; this happens for instance when $V \equiv 0$. Then replacing F with $\langle D_x \rangle F$ in (4.25) we finally obtain

$$\left\| \int_0^t e^{i(t-s)H} F(s) ds \right\|_{L^p_t L^2_y L^q_x} \lesssim \|\langle D_x \rangle F\|_{L^{\widetilde{p}'}_t L^2_y L^{\widetilde{q}'}_x}, \tag{4.26}$$

i.e., the solution of (4.16) satisfies

$$\|u\|_{L^p_t L^2_y L^q_x} \lesssim \|\langle D_x \rangle^{1/2} f\|_{L^2(\Omega)} + \|\langle D_x \rangle F\|_{L^{\widetilde{p}'}_t L^2_y L^{\widetilde{q}'}_x} \tag{4.27}$$

for all admissible couples (p,q) and $(\widetilde{p},\widetilde{q})$ with $(p,\widetilde{p}) \neq (2,2)$.

Remark 4.2. In forthcoming works we shall apply the above Strichartz estimates to investigate the existence of small global solutions for nonlinear Schrödinger and wave equations on non flat waveguides.

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